The Geometry and Algebra of Helical Tracks

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Abstract

I define the parameters of a helical track in a solenoidal magnetic field, and derive useful expressions for the algebra.

1 Introduction

In a uniform solenoidal magnetic field, the trajectory of a charged track is a helix, with the helix axis defined by the magnetic field, $\vec{B}$. We arrange our coordinate system such that $\vec{B}$ defines the z-axis. Since the sign of $\vec{B}$ can change, the direction of the z-axis is fixed with respect to the detector geometry. Figure 1 shows the $r-\phi$ projection of such a helical track, with x, y, and z forming a right-handed coordinate system.

The track can be parameterized in many ways. To start, I use the CLEO convention[**?**]. I distinguish between the *physics* parameters of the track (i.e. $\vec{p}$, $\vec{x}$, etc.) and the *geometrical* parameters. For the most part, track reconstruction is only concerned with the geometrical parameters, although of course quantities such as the momentum and energy loss are needed for calculating contributions from multiple scattering and energy loss.

A track emanating from a primary or secondary vertex is uniquely specified by 5 parameters. If the unit vector of the track is $\hat{u}$, and the unit vectors of the coordinate axes $\hat{x}$, $\hat{y}$, $\hat{z}$, then the angle ϕ is defined by $\cos\phi = \hat{u}\cdot\hat{x}$.

κ – the signed curvature of the track. The curvature is positive ($Q = 1$), if ϕ increases in moving out along the track from the point of closest approach (pca) to the z-axis. Note that geometric curvature, Q, is *not* the same as the physical curvature, q.

ϕ_0 – the $r-\phi$ direction of the track at the point of closest approach to the z-axis, i.e. $\phi = \phi_0$ at the pca.

d_0 – the signed distance between the z-axis and the point of closest approach to the z-axis. For $Q = +1$, d_0 is negative if the track circle contains the origin, positive otherwise, and *vice versa* for $Q = -1$.

z_0 – the distance in z between the point of closest approach to the z-axis and the $r-\phi$ plane. Since z_0 is measured along the z-axis, it is correspondingly signed.

$\tan\lambda$ – the tangent of the angle between the $r-\phi$ plane and a 3-D track vector, i.e. $\tan\lambda = \cot\theta = \hat{u}\cdot\hat{z}$. λ is also called the dip angle, because it measures how much the track dips out of the $r-\phi$ plane.

Other useful quantities used to describe the track are:

ρ – the radius of curvature. We define $\kappa = Q/2\rho$. The acceptance of the detector gives the minimum radius that a track must reach in order to be detected. ρ is then bounded from below, making κ bounded from above. An infinite momentum track corresponds to $\kappa =$. Not only is $|\kappa|$ bounded in $[0, Q/2\rho_{\mathrm{min}}]$, but as a nearly straight track changes sign from positive to negative, κ changes continuously from $+\epsilon$ to $-\epsilon$. The resulting numerical stability makes κ preferable to ρ. The factor of 2 in the definition is simply for algebraic convenience.

$(x_{\mathbf{c}}, y_{\mathbf{c}})$ – the coordinates of the center of the track circle. Except for initial derivations, x_{c} and y_{c} are not used for the same reason as ρ.

(x_0, y_0) – the coordinates of the oint of closest approach in the $r-\phi$ plane.

$\rho_{\mathbf{c}}$ – the distance in the $r-\phi$ plane from the center of the track circle to the z-axis.

$\phi_{\mathbf{c}}$ – the azimuth of the track circle.

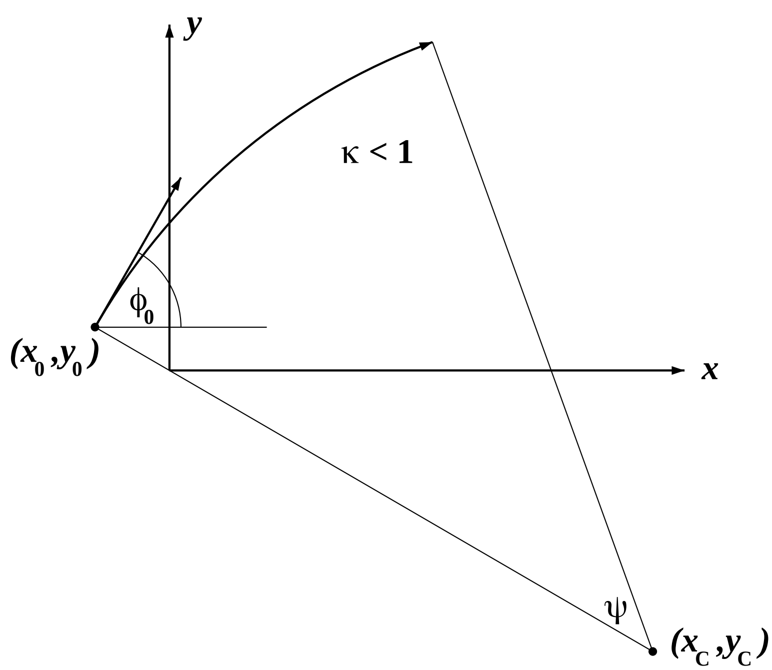

Figure 1: The $r - \phi$ projection of a helical track

ψ – the angle through which the track rotates in moving from the pca to any point (x, y) on the track.

From the definitions, and by inspection, we can write down expressions between these other track quantities:

$$
\begin{aligned}
\phi_C &= \phi_0 + Q\pi/2 && (1)\\
\rho_C &= \rho + Qd_0 && (2)\\
x_0 &= -d_0 \sin\phi_0 && (3)\\
y_0 &= d_0 \cos\phi_0 && (4)\\
d_0 &= y_0 \cos\phi_0 - x_0 \sin\phi_0 && (5)
\end{aligned}
$$

The relation between the geometrical parameters and the physical parameters then follows. For a track of momentum p, charge q, and polar angle θ, the momentum, signed by the physical charge q, in units of GeV/c is given by:

$$p\cos\theta = p_T = \frac{aB}{\kappa} \quad (6)$$

where B is the signed (axial) magnetic field in Tesla, p_T is the transverse momentum in GeV/c, and the constant a is:

$$a = \frac{1}{2}\, c10^{-9} \simeq 0.15\mathrm{GeV} \cdot \sec \cdot \mathrm{T}^{-1} \quad (7)$$

2 Cartesian Parameterization

It is often more convenient to use a Cartesian parameterization of the track. From the geometry of Figure 1, any point (x, y) on the circular track is given by:

$$(x - x_{\rm C})^2 + (y - y_{\rm C})^2 = \rho^2 \tag{8}$$

Multiplying out and dividing by $2Q\rho$ gives:

$$\kappa r^2 + \alpha x + \beta y + \gamma = 0 \tag{9}$$

where $r^2 = x^2 + y^2$ and:

$$\begin{aligned}
\kappa &= \frac{Q}{2\rho} &(10)\\
\alpha &= \frac{-Qx_{\rm C}}{\rho} &(11)\\
\beta &= \frac{-Qy_{\rm C}}{\rho} &(12)\\
\gamma &= \frac{Q(x_c^2 + y_c^2 - \rho^2)}{2\rho} &(13)
\end{aligned}$$

which introduces the four parameters, $\kappa, \alpha, \beta, \gamma$, with only 3 independent ones. The constraint equation then defines another parameter ξ, such that:

$$\xi^2 = \alpha^2 + \beta^2 = 1 + 4\kappa\gamma \tag{14}$$

At the expense of adding a quadratic constraint equation, we have linearized the circle equation, which will lead to many computational savings.

We can now trivially derive relations between the 2 parameterizations:

$$\begin{aligned}
\rho_c^2 &= \rho^2(\alpha^2 + \beta^2) = \rho^2\xi^2 &(15)\\
\xi &= 1 + 2\kappa d_0 &(16)\\
\alpha &= \xi \sin\phi_0 &(17)\\
\beta &= -\xi\cos\phi_0 &(18)\\
\gamma &= d_0(1 + \kappa d_0) &(19)
\end{aligned}$$

with the inverse relationships:

$$\begin{aligned}
d_0 &= \frac{(\xi - 1)}{2\kappa} = \frac{2\gamma}{(\xi + 1)} &(20)\\
\phi_0 &= \tan^{-1}\left(\frac{\alpha}{-\beta}\right) = \texttt{atan2}(\alpha, -\beta) &(21)
\end{aligned}$$

(using the `C` function `atan2` to return a result in the range $[-\pi, pi]$). The coordinates at the origin are:

$$\begin{aligned}
x_0 &= \frac{-2\alpha\gamma}{\xi(\xi + 1)} &(22)\\
y_0 &= \frac{-2\beta\gamma}{\xi(\xi + 1)} &(23)
\end{aligned}$$

3 Intersection of Track with a Circle

Consider the intersection of a track with a circle of radius R centered at the origin, such as a drift chamber tracking layer. If the curvature is too large, the track curls before even reaching the circle. At some radius, $R_{\rm curl}$, the track just grazes the circle at one point. At smaller R (or smaller κ, the track intersects the circle once on the outward path, and a second time on the inward path. Let (x, y) be the coordinates of the intersection point. Then the simultaneous equations for the track and the circle are:

$$\kappa r^2 + \alpha x + \beta y + \gamma = 0 \tag{24}$$
$$x^2 + y^2 = R^2 \tag{25}$$

solving for x and y:

$$x\xi^2 = -\alpha(\kappa R^2 + \gamma) \mp \beta \left(R^2\xi^2 - (\kappa R^2 + \gamma)^2\right)^{1/2} \tag{26}$$
$$y\xi^2 = -\beta(\kappa R^2 + \gamma) \pm \alpha \left(R^2\xi^2 - (\kappa R^2 + \gamma)^2\right)^{1/2} \tag{27}$$

where the first sign in the $\mp$ ($\pm$) refers to the outward branch of the track, and the second sign to the inward branch. The solutions for x and y become single-valued at the minimum and maximum values of R, which are found by equating the argument of the square root to zero. The minimum radius, $R_{\rm min}$, is just d_0; the maximum radius, $R{\rm max}$, is $R_{\rm curl}$:

$$R_{\rm min} = |d_0| = \left|\frac{(\xi - 1)}{2\kappa}\right| = \left|\frac{2\gamma}{(\xi + 1)}\right| \tag{28}$$
$$\tag{29}$$
$$R_{\rm max} = R_{\rm curl} = \left|\frac{(\xi + 1)}{2\kappa}\right| = \left|\frac{2\gamma}{(\xi - 1)}\right| \tag{30}$$

The coordinates at $R_{\rm curl}$ are:

$$x_{\rm curl} = \frac{-\alpha(\xi + 1)}{2\xi\kappa} \tag{31}$$
$$y_{\rm curl} = \frac{-\beta(\xi + 1)}{2\xi\kappa} \tag{32}$$
$$\tag{33}$$

Rearranging to find the value of κ corresponding to $R_{\rm curl}$:

$$\kappa = \frac{QR + \gamma}{R^2} \tag{34}$$

To find the $r - \phi$ direction cosines at (x, y), we can differentiate equation (9):

$$\kappa \left(2x + 2y\,\frac{dy}{dx}\right) + \alpha + \beta\,\frac{dy}{dx} = 0 \tag{35}$$

and rearranging:

$$\frac{dy}{dx} = -\frac{(2\kappa x + \alpha)}{(2\kappa y + \beta)} \tag{36}$$

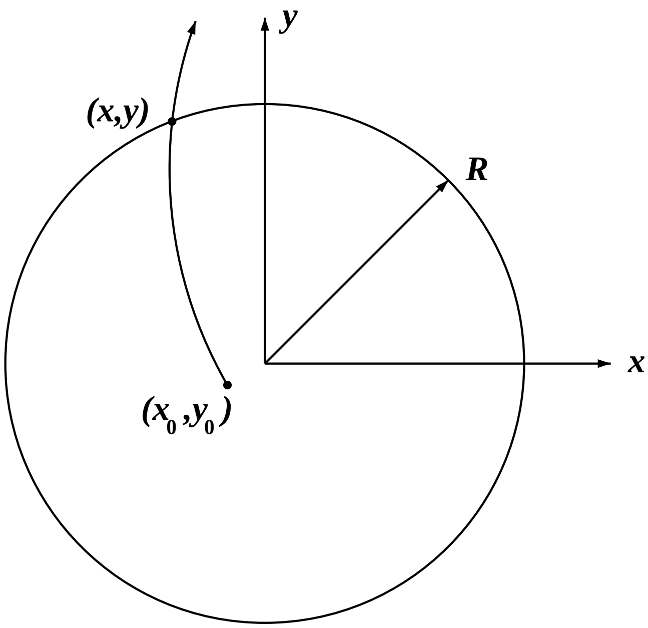

Figure 2: The intersection of a track with a circle of radius R

Denoting the x and y components of the $r-\phi$ direction cosines by x' and y', it follows that:

$$x' = -(2\kappa y + \beta) \tag{37}$$

$$y' = (2\kappa x + \alpha) \tag{38}$$

It follows automatically that $x'^2 + y'^2 = 1$. Moreover, at the point $x_0, y_0)$, x' and y' become $\cos\phi_0$ and $\sin\phi_0$ respectively. Hence the signs and normalization of equations 37-38 are correct.

We can use equations 37-38 to derive equations for the entrance angle of a track into a cylinder of radius R. The unit vector of the cyclinder at the point (x, y) is radial, and using the unit vector of the track:

$$\hat{u} = (x', y') \tag{39}$$

$$\hat{r} = (x, y)/R \tag{40}$$

denoting the angle between $\hat{u}$ and $\hat{r}$ by θ, it follows that:

$$\sin\theta = \hat{u} \times \hat{r} = -\frac{\kappa R^2 - \gamma}{R} \tag{41}$$

$$\cos\theta = \hat{u} \cdot \hat{r} = \frac{(\alpha y - \beta x)}{R} = \pm\frac{1}{R}\left(R^2\xi^2 - (\kappa R^2 + \gamma)^2\right)^{1/2} \tag{42}$$

The meaning of $\pm$ for $\cos\theta$ is the same as in equations 26-27, showing explicitly the correct sign for tracks on the outward (inward) part of the trajectory. The sign of $\cos\theta$ from equation 42, determined from $(\alpha y - \beta x)$ can then be used to determine the branch (inward or outward) of a track.

4 Distance of a Point from Track

Consider an arbitrary point, (x, y), such as in Figure 4. Define d to be the perpendicular distance from (x, y) to the track, with a sign defined by:

$$\begin{aligned} d &= Q\left\{\left[(x-x_C)^2+(y-y_C)^2\right]^{1/2}-\rho\right\} \\ &= \frac{1}{2\kappa}\left\{\left[(\alpha+2\kappa x)^2+(\beta+2\kappa y)^2\right]^{1/2}-1\right\} \\ &= \frac{1}{2\kappa}\left\{\left[1+4\kappa(\kappa r^2+\alpha x+\beta y+\gamma)\right]^{1/2}-1\right\} \\ &= \frac{\sqrt{1+4\kappa\delta}-1}{2\kappa} \end{aligned} \tag{43}$$

where:

$$\delta = \kappa r^2 + \alpha x + \beta y + \gamma \tag{44}$$

Clearly, if (x, y) lies on the track, $\delta = 0$, and so $d = 0$. If (x, y) is the origin, then $\delta = \gamma$, and:

$$d = \frac{\sqrt{1+4\kappa\gamma}-1}{2\kappa} \tag{45}$$

But from equation 20,

$$d_0 = \frac{(\xi-1)}{2\kappa} = \frac{\sqrt{1+4\kappa\gamma}-1}{2\kappa} \tag{46}$$

In other words, the sign used in equation 43 is consistent with our definition of d_0.

If the point is not too far from the track, then we can expand the square root, and approximate equation 43 to:

$$d = \delta(1-\kappa\delta)+\ldots \tag{47}$$

5 Determination of Track Parameters from Three Points

The $r-\phi$ parameters of a circular track can be uniquely determined by specifying three points. Let these points be (x_1, y_1), (x_2, y_2), and (x_3, y_3). Using equation (9), we can write down 3 equations for the 3 unknows:

$$\kappa r_1^2 + \alpha x_1 + \beta y_1 + \gamma = 0 \tag{48}$$

$$\kappa r_2^2 + \alpha x_2 + \beta y_2 + \gamma = 0 \tag{49}$$

$$\kappa r_3^2 + \alpha x_3 + \beta y_3 + \gamma = 0 \tag{50}$$

The curvature, κ, is independent of the actual coordinate origin, although the other parameters are not. Therefore, we can shift the origin to (x_1, y_1) to calculate κ. In this new coordinate system, $\alpha \to \alpha'$, $\beta \to \beta'$, and $\gamma \to \gamma' = 0$. The equations for points 2 and 3 are then:

$$\kappa r_{21}^2 + \alpha' x_{21} + \beta' y_{21} = 0 \tag{51}$$

$$\kappa r_{31}^2 + \alpha' x_{31} + \beta' y_{31} = 0 \tag{52}$$

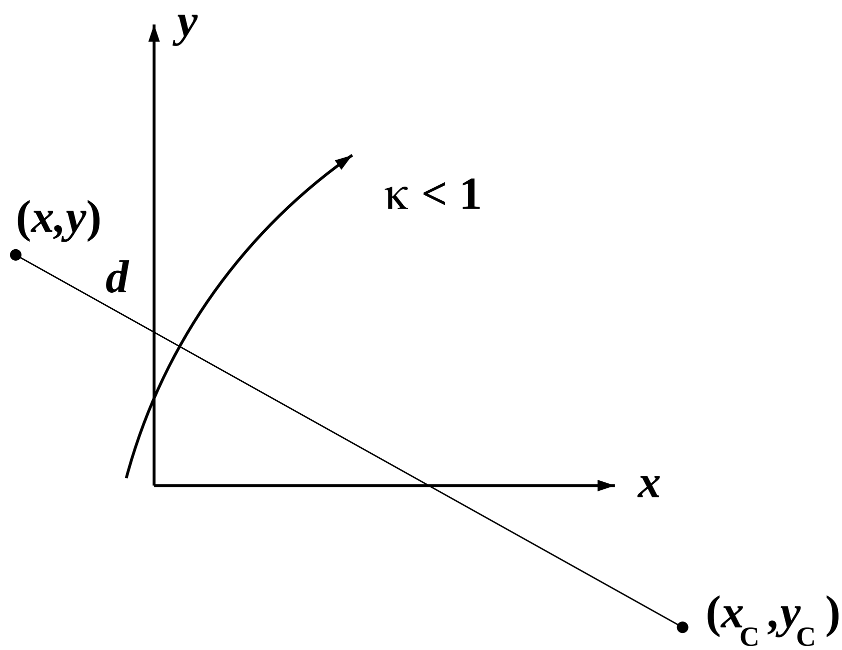

Figure 3: Distance from Point to Track

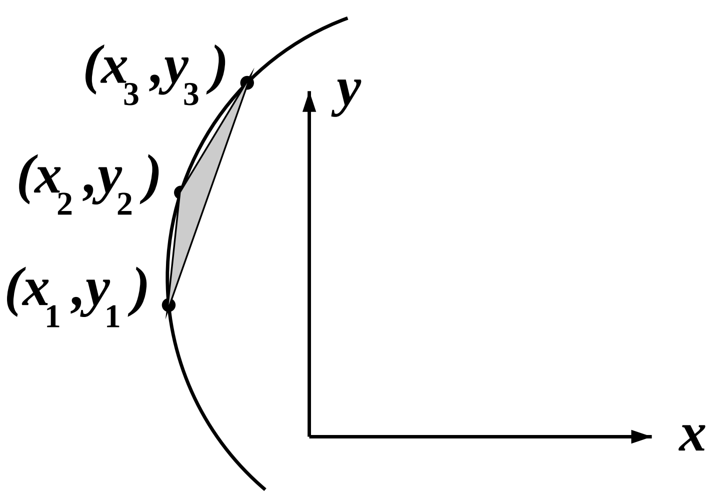

Figure 4: Determination of Track Parameters from 3 points

where x_{21} is measured from point 2 to point 1, and similarly for the other quantities. Solving for α', β':

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \frac{-\kappa}{\Delta} \begin{pmatrix} y_{31} & -y_{21} \\ -x_{31} & x_{21} \end{pmatrix} \begin{pmatrix} r_{21}^2 \\ r_{31}^2 \end{pmatrix} \tag{53}$$

where:

$$\Delta = x_{21}y_{31} - y_{21}x_{31} = (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1) = \Delta_{12} + \Delta_{23} + \Delta_{31} \tag{54}$$

and $\Delta_{ij} = x_i y_j - x_j y_i$. Δ is just the area of the triangle defined by the 3 points, and shown in Figure 5, since if $\vec{\ell}_{ij}$ is the vector from point i to point j:

$$\vec{\ell}_{ij} = (x_j - x_i, y_j - y_i) \tag{55}$$

it follows that:

$$\Delta = \left(\vec{\ell}_{12} \times \vec{\ell}_{13}\right)_z = \ell_{12} \cdot \ell_{13} \sin\theta \tag{56}$$

A positive Δ means that the angle θ is positive, so the curvature of the track, κ, has the same sign as Δ, and if $\Delta = 0$, then $\kappa = 0$.

Since $\gamma' = 0$, $\alpha'^2 + \beta'^2 = 1$, we get an expression for κ:

$$\frac{\kappa^2}{\Delta^2} \left[\left(y_{31}r_{21}^2 - y_{21}r_{31}^2\right)^2 + \left(-x_{31}r_{21}^2 + x_{21}r_{31}^2\right)^2\right] = 1 \tag{57}$$

and simplifying:

$$\frac{\kappa^2}{\Delta^2} r_{31}^2 r_{21}^2 \left[r_{21}^2 - 2\left(x_{31}x21 + y_{31}y21\right) + r_{31}^2\right] = 1 \tag{58}$$

The middle term is just the dot product:

$$x_{31}x21 + y_{31}y21 = \vec{\ell}_{12} \cdot \vec{\ell}_{13} = r_{21}r_{31}\cos\theta \tag{59}$$

and from the cosine rule, $r_{21}^2 - 2r_{21}r_{31}\cos\theta + r_{31}^2 = r_{32}^2$, so finally we get the expression for κ:

$$\kappa = \frac{\Delta}{r_{12}r_{23}r_{31}} \tag{60}$$

which from the previous argument, also has the correct sign. That is, κ is the area of the triangle defined by the 3 points, divided by the lengths of the 3 sides.

Using this expression for κ, (or κ/Δ), we can immediately solve for α' and β', as given by equation 53. To calculate γ, we can find the distance from the track to the origin in the *original* coordinate system using equation 43. In the translated coordinate system, the old origin is at $(-x_1, -y_1)$, so δ in equation 43 is:

$$\delta = \kappa r_1^2 - \alpha' x_1 - \beta' y_1 \tag{61}$$

since $\gamma' = 0$. But in the original coordinate system, $\gamma = \delta$, since δ is calculated from the origin. We have therefore solved for γ. If we need α and β, we can take equations 48-49, using κ and γ:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{-1}{\Delta_{12}} \begin{pmatrix} y_2 & -y_1 \\ -x_2 & x_1 \end{pmatrix} \begin{pmatrix} \kappa r_1^2 + \gamma \\ \kappa r_2^2 + \gamma \end{pmatrix} \tag{62}$$